A Pro-Competitive Effect of Joint Bidding in Multi-Unit Uniform Price Auction with Asymmetric Bidders

Jinsoo Bae *

Department of Economics, The Ohio State University

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Abstract

This paper studies a pro-competitive effect of joint bidding in multi-unit uniform price auctions where bidders have private values and demand different quantities of units. We analyze a simple model with three identical items for sale, two small bidders each demanding a single unit, and a big bidder demanding two units. We show that joint bidding of the two small bidders, which recovers the symmetry of bidders, enhances competition among the bidders and increases efficiency and revenue of the auction.

Keywords: Joint bidding, Multi-unit auction, Asymmetric bidders

JEL Classification: D44,D43

*Correspondence to: Department of Economics, The Ohio State University, 410 Arps Hall, 1945 N. High St. Columbus OH, 43210, USA. *E-mail address*: bae.146@osu.edu

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1 Introduction

Studies on anti- and pro-competitive effects of joint bidding have been a subject for public policy debates over the past few decades. Ever since Markham (1970), researchers have been concerned about the anti-competitive effects of joint bidding, as it reduces the number of bidders, thus reducing competition. However, pro-competitive arguments of joint bidding have been reported as well. First, joint bidding serves as a vehicle for circumventing capital constraints, thus encouraging entry by small bidders (Hendricks and Porter, 1992). Second, when auctioned objects are risky, joint bidding allows bidders to share the risk, leading to more aggressive bidding (Millsaps and Ott, 1985). Third, in common value auctions, joint bidding allows bidders to pool their information to have more precise estimates of the value of the item, thus mitigating the winner's curse, leading to more aggressive bids (DeBrock and Smith, 1983; Krishna and Morgan, 1997). In this paper, I demonstrate a pro-competitive effect of joint bidding based on mitigating demand reduction rather than capital constraint, risk, and information pooling.

Previous studies showing pro-competitive effects of joint bidding focus on single-unit auctions. In contrast, the current paper reports a pro-competitive effect of joint bidding in *multi-unit* uniform price auctions where bidders have private values and demand different numbers of units. I consider a uniform price auction where three identical items are sold, and a big bidder demands two units and two small bidders each demand one unit. In equilibrium, the small bidders submit bids equal to their values and the big bidder reduces demand for his second unit to get the first unit at a cheaper price. Now suppose the two small bidders merge and bid jointly[1]. The merger has two effects: first, the merged bidders now also reduce demand for their second unit, which adversely affects the seller's revenue. Second, their reduction in demand for the second unit encourages the big bidder to bid more for his

[1]So that they behave as a single bidder demanding up to two units.

second unit, as he now has a chance of winning the second unit at an advantageous price. When the second effect dominates the first effect, it increases the clearing price and the seller's revenue.

The asymmetry among bidders in the model plays an important role as the pro-competitive effect of joint bidding is attributed to the fact that the joint bidding alters an asymmetric competition structure into a symmetric one. Thus, joint bidding among symmetric bidders would not have such a pro-competitive effect. Levin (2004) analyzes the competitive effect of joint bidding of symmetric bidders who originally demand a single unit then, after they are randomly paired up, demand up to two units with joint bidding. In his model, no demand reduction exists without joint bidding so joint bidding only increases monopsony power and strengthens anti-competitive behavior.

2 The Model

Suppose a seller auctions off *three* identical items in a *uniform price auction* and there are three bidders 1,2 and 3. Bidder 1 demands up to two units (big bidder) and the other bidders each demand a single unit (small bidder), so the total demand is four units. All valuations for the items are *independently* drawn from a CDF, $F(v)$, whose support is $[0, 1]$. Bidder 1's valuations are $\{\hat{v}_1, v_1\}$ where $\hat{v}_1 \geq v_1$ (note that bidder 1's valuations are two independent random draws from $F(v)$) and bidder 2 and 3's valuations are v_2 and v_3, respectively

2.1 Without Joint Bidding

In the auction, bidder 1 submits bids $\{\hat{b}_1^I, b_1^I\}$ where $\hat{b}_1^I \geq b_1^I$ and bidder 2 and 3 respectively submit b_2^I and b_3^I [2]. Without joint bidding, the three highest bids win the items and the lowest bid sets the price. Bidders 2 and 3 have a (weakly) dominant strategy of bidding

[2] Superscript I means bids without joint bidding (**I**ndividually bidding)

their values ($b_2^I(v_2) = v_2$, $b_3^I(v_3) = v_3$). Bidder 1 is guaranteed to get his first unit, so the question is how much he bids on the second unit. Thus, the equilibrium is characterized by the optimal bid function $b_1^I(v_1)$, given the other bidders submit bids equal to their values.

Since bidder 1 wins two units when $b_1^I(v_1) > min(v_2, v_3)$ and one unit otherwise, it is convenient to define $v_l = min(v_2, v_3)$. Since v_2 and v_3 are independent, v_l is a random draw from a CDF, $H(v) = 1 - (1 - F(v))^2$. The expected payoff of bidder 1 who bids b_1^I for his second unit is

$$\Pi(b_1^I; \hat{v}_1, v_1) = \hat{v}_1 + \int_0^{b_1^I} (v_1 - 2v_l)h(v_l)dv_l + (-b_1^I)(1 - H(b_1^I))$$

By differentiating the expected utility with respect to b_1^I, I get the following necessary condition for the optimal bid $b_1^I(v_1)$:

$$\frac{d\Pi}{db_1^I} = (v_1 - b_1^I)h(b_1^I) - [1 - H(b_1^I)] \leq 0 \tag{1}$$

with strict inequality only if the optimal bid is the corner solution $b_1^I(v_1) = 0$.

From the necessary condition, it is clear that bidding the value of the second unit is never optimal ($\frac{d\Pi}{db_1^I}|_{b_1^I = v_1} < 0$), so bidder 1 must exercise demand reduction for his second unit.

2.2 With Joint Bidding

Now suppose bidders 2 and 3 merge into a bid consortium and bid jointly. The bid consortium demands up to two units and the valuations for the two units follow the original valuations of the two bidders $\{v_2, v_3\}$. For notational convenience, I denote the higher one of the two values $\hat{v}_c$ and the lower one v_c, so the valuations of the bid consortium are $\{\hat{v}_c, v_c\}$, where $\hat{v}_c \geq v_c$. Note that v_c is a random draw from a CDF, $H(v) = 1 - (1 - F(v))^2$ since it is the lower of two independent draws $\{v_2, v_3\}$ from $F(v)$.

In the auction, bidder 1 submits bids $\{\hat{b}_1^J, b_1^J\}$ where $\hat{b}_1^J \geq b_1^J$ and the bid consortium bids $\{\hat{b}_c^J, b_c^J\}$ where $\hat{b}_c^J \geq b_c^J$ [3]. The three highest bids win the items and the lowest bid sets the price. Note that now there are two symmetric bidders (bidder 1 and bid consortium) who both demand two units, with each bidder guaranteed to get his first unit. Therefore, a symmetric Bayesian Nash equilibrium is obtained by characterizing a symmetric bid function for the second unit, $b^J(v)$. The expected payoff of bidder 1 who bids b_1^J for his second item is:

$$\Pi(b_1^J, \hat{v}_1, v_1) = \hat{v}_1 + \int_0^{(b^J)^{-1}(b_1^J)} (v_1 - 2b^J(v_c))h(v_c)dv_c + (-b_1^J)[1 - H((b^J)^{-1}(b_1^J))]$$

By differentiating the expected payoff with respect to b_1^J and using that it is optimal to bid $b^J(v)$ at an equilibrium, I get the following necessary condition for an equilibrium bid function [4].

$$(v_1 - b^J(v_1))h(v_1) - (b^J)'(v_1)(1 - H(v_1)) = 0$$

Solving this ODE gives the equilibrium bid function, which is a special case of Theorem 6 of Engelbrecht-Wiggans and Kahn (2002)

$$b^J(v_1) = v_1 - \int_0^{v_1} \frac{1 - H(v_1)}{1 - H(x)} dx \tag{2}$$

From the bidding function (2), it is clear that bidder 1 and the bid consortium exercise demand reduction.

Without joint bidding, only bidder 1 reduces his demand, but with the joint bidding, all bidders reduce their demands. So does this mean that the joint bidding reduce the seller's revenue? The answer is not necessarily. Demand reduction of the merged bidders mitigate big bidder's incentive to submit a low bid for the second unit as the big bidder now have

[3] Superscript J denotes bids with **J**oint bidding

[4] Note that $b^J(v) = 0$ never occurs with the joint bidding. If $b^J(v) = 0$ for any interval, then bidding slightly more than 0 will break the tie and ensures the second unit, breaking the equilibrium.

a chance of winning the second unit at a cheaper price. I will show an example where the joint bidding mitigate the demand reduction of bidders 1 (i.e $b_1^I(v_1) < b^J(v_1)$) and the seller's revenue increases.

2.3 An Example of the Pro-Competitive Effect of Joint Bidding

Example 1. When $F(v) = 1 - \sqrt{1-v}$, so $H(v) = v$, bidder 1's optimal bid without joint bidding is $b_1^I(v_1) = 0$ and the optimal bid with the joint bidding is $b^J(v_1) = v_1 - (1-v_1)ln\frac{1}{1-v_1}$. Therefore, the seller's revenue is greater with the joint bidding.

To see this, note that the optimal bid of bidder 1 without the joint bidding, $b_1^I(v_1)$, is characterized by (1):

$$\begin{aligned}\frac{d\Pi}{db_1^I} = (v_1 - b_1^I)h(b_1^I) - [1 - H(b_1^I)] &= (v_1 - b_1^I) - (1 - b_1^I) \\ &= v_1 - 1 < 0, \quad \text{for all} \quad v_1 \in [0,1)\end{aligned}$$

Thus, $b_1^I(v_1) = 0$. On the other hand, $b^J(v_1) = v_1 - (1 - v_1)ln\frac{1}{1-v_1} > 0$ is obtained by (2).

Without the bid consortium, bidder 1 completely reduces his demand for his second unit, setting the clearing price at 0, hence the seller earns nothing. Since the two small bidders submit bids equal to their valuations and their valuations are likely to be high ($F(v)$ has higher probability density at higher values), it is more profitable for bidder 1 to avoid the competition for his second unit with the small bidders and set a lower price for his first item. However, with joint bidding, bidder 1 finds it worth submitting a positive bid for his second unit. This is because the demand reduction of the bid consortium for their second unit gives a chance for a bidder 1 to get his second unit at a favorable price.

3 Environments for the pro-competitive effect

In this section, I analyze the environments where the pro-competitive effect is likely to appear. Due to asymmetry of bidders and non-existence of explicit bid function of non-merger case, $b_1^I(v_1)$, it is formidable [5] to generalize conditions where the seller's (expected) revenue increases with the joint bidding. However, since the seller's revenue is closely related to the magnitudes of demand reduction with and without merger, I will focus the environments where the demand reduction mitigated by the merger (i.e. $b_1^I(v_1) < b^J(v_1)$). In particular, when bidder 1 completely reduces his demand under non-merger, $b_1^I(v_1) = 0 < b^J(v_1)$, the seller's revenue will unambiguously increase with the merger.

3.1 When bidder 1 has small realization of v_1

Proposition 1. If $\lim_{b\to 0+} h(b) < \infty$, there exists $v^* \in (0,1)$ below which $b_1^I(v) = 0$. On the contrary, $b^J(v) > 0$ regardless of distribution of values.

Proof. Since $\lim_{b\to 0+} h(b) < \infty$, $h(b) < \infty$ for all $b < v \in (0,1)$. Bidder 1 with value v will completely reduce his demand for the second unit if $(v-b)h(b) - [1-H(b)] < 0$ for all $b \leq v$ [6]

$$\sup_{b\leq v}\,[(v-b)h(b) - [1-H(b)]] = v\sup_{b\leq v} h(b) - [1-H(v)]$$

$$v\sup_{b\leq v} h(b) - [1-H(v)] < 0 \quad \text{when} \quad v < \frac{1-H(v)}{\sup_{b<v} h(b)}$$

v is increasing from 0 and $\frac{1-H(v)}{\sup_{b<v} h(b)}$ decreasing from a positive number since $\lim_{b\to 0+} h(b) < \infty$. Therefore, there is $v^* \in (0,1)$ below which $v\sup_{b\leq v} h(b) - [1-H(v)] < 0$.

[5] one may use the argument of being efficient is being optimal.. (Ausbel and Crampton Paper,1999) This is just a conjecture for now but if this works out, the ideas can even extended to even more number of bidders.

[6] Bidding above value is dominated, so I only need to consider $b \leq v$

With the merger, as discussed in 2.3, a symmetric equilibrium cannot have a bid function that has an interval where full complete reduction arises. In particular, there is no such complete demand reduction even with a small value. To show this I set up a lemma that will be frequently used in this paper.

Lemma 1. Equation (2), the necessary condition for $b^J(v)$, can be rewritten as

$$(v_1 - b)g(b) - [1 - G(b)] = 0, \quad \text{where} \quad G(b) = H((b^J)^{-1}(b)) \tag{3}$$

Proof. See the Appendix. □

Now, I will show that $\lim_{b \to 0+} g(b) = \infty$, thus $b^J(v)$ does not subject to proposition 1.

$$g(b) = G'(b) = H'((b^J)^{-1}(b)) =$$

$$h((b^J)^{-1}(b))\frac{1}{(b^J)'((b^J)^{-1}(b))} = h((b^J)^{-1}(b))\frac{1}{h((b^J)^{-1}(b))\int_0^{(b^J)^{-1}(b)} \frac{1}{1-H(x)}dx}$$

$$= \frac{1}{\int_0^{(b^J)^{-1}(b)} \frac{1}{1-H(x)}dx}, \quad \text{thus} \quad \lim_{b \to 0+} g(b) = \infty$$

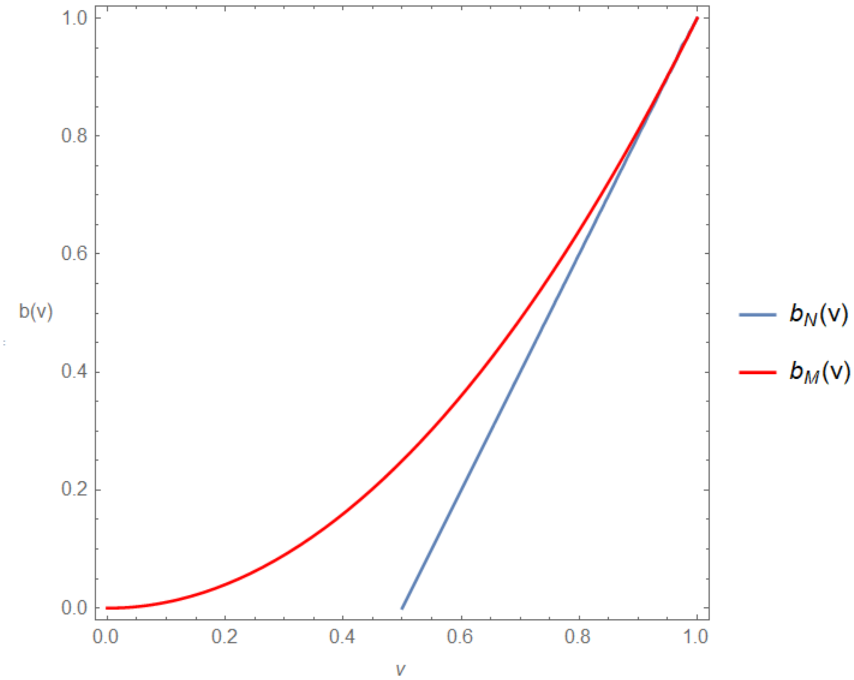
□

Without joint bidding, if bidder 1's realized valuation for this second unit is so small, the benefit of increasing the probability of winning the second item by bidding more is so small, its better to give up the competition for the second item. The only case where it is still worth competing for the second item with a small realized value is when the probability distribution $h(b)$ has unbounded density for small values, so the competing bidders also very likely to receive small realized values.

Example 2. Suppose $F(x) = x$, so $H(x) = 1 - (1-x)^2$ and $h(x) = 2(1-x) < \infty$. As proposition 2, $b_N(v) = 0$ when $v < 0.5$ and $b_1^I(v) = 2v - 1$, $v \geq 0.5$. Moreover, $b_1^I(v) < b^J(v)$

for all $v \in [0,1)$ (see figure 1) and the seller's expected revenue increases from 0.15 to 0.2 with the merger.

Figure 1: Bid functions



Proof. For the revenue part, see the Appendix □

3.2 When $H(v)$ has a high expected value

Proposition 2. If $H(b) \leq b$ (graphically, $H(b)$ is below 45 degree line), then $b_1^I(v) = 0$ for all v.

Proof. See the Appendix □

Intuitively, $H(b) < b$ implies that, in expectation, a random draw from $H(b)$ is likely to be high since $E(v) = \int_0^1 [1 - H(v)]dv$ [7]. Thus, bidder 1 is likely to face rivals with high values.

[7] For non-negative random variable, integration above a CDF is the expected value

In the non-merger case, bidder 2 and 3 bid their values. Therefor, bidder 1 can better of by giving up competing for the second unit and set the price at 0 regardless of the realized value of bidder 1.

Corollary 3. If $H(b) \leq b$, sellers revenue under non-merger is 0. Thus, the seller's revenue unambiguously increases with the merger.

Example 3. When $H(x) = x^a$ and $a \geq 1$, $b_N(v) = 0$. On the contrary, when $H(x) = x^a$ and $a < 1$, $b^I(v) > 0$ for all $v > 0$.

Proof. By proposition 2, the former statement is true. For the latter one, see the appendix

□

3.3 The General Condition for $b_1^I(b) < b^J(b)$

Proposition 4. A sufficient condition $b_1^I(b) < b^J(b)$ for all $v \in [0,1]$ is $\frac{h(b)}{1-H(b)} < \frac{g(b)}{1-G(b)}$ for $b \in [0,1]$, where $G(b) = H((b^J)^{-1}(b))$.

Proof. The necessary conditions of optimal bidding in non-merger (1) and merger (3) cases can be re-written as

$$(v-b)\frac{h(b)}{[1-H(b)]} - 1 = 0 \tag{4}$$

$$(v-b)\frac{g(b)}{[1-G(b)]} - 1 = 0 \tag{5}$$

Suppose all b that satisfy $(v-b)\frac{h(b)}{[1-H(b)]} - 1 = 0$ also satisfy $(v-b)\frac{g(b)}{[1-G(b)]} - 1 > 0$. This is a sufficient condition for $b^J(v) > b_1^I(v)$.

$$(v-b)\frac{h(b)}{[1-H(b)]} - 1 = 0$$

$$(v-b)\frac{g(b)}{[1-G(b)]} - 1 = \frac{[1-H(b)]}{h(b)}\frac{g(b)}{[1-G(b)]} - 1$$

From this, $\frac{g(b)}{[1-G(b)]} > \frac{h(b)}{[1-H(b)]}$ for all b is a sufficient condition for $b^J(v) > b_1^I(v)$.

Using $G(b) = H((b^J)^{-1}(b))$, the sufficient condition can be re-written as

$$\frac{1}{[1 - H((b^J)^{-1}(b))] \int_0^{(b^J)^{-1}(b)} \frac{1}{1-H(x)} dx} > \frac{h(b)}{[1 - H(b)]} \quad \text{For all} \quad b \in [0.1]$$

□

The sufficient condition means if hazard rate of $G(.)$ is greater than hazard rate of $H(.)$, then $b_1^I(v) < b^J(v)$. Intuitively, hazard rate means the instantaneous probability that bidding slightly more leads to winning when the original bid was a losing bid. (4) and (5) means that, conditional on losing the second unit, the marginal benefit of bidding slight more is profit margin of second unit multiplied by hazard rate and the marginal cost is a direct loss from setting higher price for the first unit. $\frac{g(b)}{[1-G(b)]} > \frac{h(b)}{[1-H(b)]}$ implies that the marginal benefit is always higher in the merger case, thus, bidder 1 must bid more in the merger case.

Corollary 5. For every distribution $H(v)$, there is v^{**} below which $b_1^I(v) < b^J(v)$

Proof. I show that there exists v^{**} below which the following inequality must hold.

$$\frac{h(b)}{1 - H(b)} < \frac{g(b)}{1 - G(b)}$$

First of all, $H(b) < G(b)$ since $G(b) = H((b^J)^{-1}(b))$ and $b < (b^J)^{-1}(b)$. Second, since $H(b) < G(b)$ and both of them are CDFs, there must be a interval $[0, v^{**}]$ where $h(b) < g(b)$. Thus the above inequity must hold for any $b < v^{**}$. □

4 Efficiency, Incentive to Merge, Generalizability

In this model, the efficiency of the auction unambiguously increases with the joint bidding of the small bidders. Without joint bidding, the big bidder avoids competition for his second unit, occasionally losing the second unit even though his value for that unit is greater than the values of bidder 2 or 3. With joint bidding, the bidder with a higher value for the second unit always wins that unit, resulting in increased efficiency. The increased efficiency in this model implies that a properly designed merger that recovers symmetry among bidders can increase social welfare as well as the seller's revenue.

A limitation of this model is that the smaller bidders may not have an incentive to bid jointly as it may reduce their profits. For instance, in example 1, the interim expected profits of smaller bidders given the valuations are $\hat{v}_c$ and v_c, while the expected joint profit of bidding jointly is $\hat{v}_c + \frac{1}{2}v_c^2$ [8]. However, if participating in the auction is costly, the smaller bidders can have an incentive to bid jointly. Suppose the participating cost is v_c. Then without joint bidding, both small bidders enter the auction and the expected profits are $\hat{v}_c - v_c$ and 0. With joint bidding, on the contrary, the bid consortium's expected joint profit is $\hat{v}_c + \frac{1}{2}v_c^2 - v_c$, which is greater than the sum of profits in individual bidding.

Future research can investigate under what conditions that the pro-competitive effect holds. It is straightforward to show that the pro-competitive effect holds for any distribution that first-order stochastically dominates the distribution in example 1. In general, however, comparing expected revenue of the seller with and without joint bidding is a difficult task due to the asymmetry of the bidders and the implicit form of the bid function $b_1^I(v_1)$.

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[8] Recall that the higher of $\{v_2, v_3\}$ is $\hat{v}_c$ and the lower is v_c

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A Appendix

A.1 Symmetry Equilibrium with Joint Bidding, $b_M(v)$

The necessary condition of the symmetric equilibrium is

$$(v - b(v))h(v) - b'(v)(1 - H(v)) = 0$$

Solving the ODE,

$$b'(v) + \frac{h(v)}{1 - H(v)}b(v) = \frac{h(v)}{1 - H(v)}v$$
$$\mu(v)b'(v) + \mu(v)\frac{h(v)}{1 - H(v)}b(v) = \mu(v)\frac{h(v)}{1 - H(v)}v$$

Assuming $\mu(v)\frac{h(v)}{1-H(v)} = \mu'(v)$

$$\mu(v)b'(v) + \mu(v)'b(v) = \mu(v)\frac{h(v)}{1 - H(v)}v$$
$$\mu(v)b'(v) = \int_0^v \mu(x)\frac{h(x)}{1 - H(x)}xdx + c$$
$$b(v) = \frac{\int_0^v \mu(x)\frac{h(x)}{1-H(x)}xdx + c}{\mu(v)}$$

Solve $\mu(v)$

$$\frac{\mu(v)}{\mu'(v)} = \frac{h(v)}{1 - H(v)}$$
$$ln(\mu(v)) = \int_0^v \frac{h(x)}{1 - H(x)}dx + k$$
$$\mu(v) = k' \exp(\int_0^v \frac{h(x)}{1 - H(x)}dx)$$
$$\mu(v) = k' \exp(-ln(1 - H(v)) = k'\frac{1}{1 - H(v)}$$

Plug in $\mu(v)$ above bidding function $b(v)$

$$b(v) = (1 - H(v)) \left[\int_0^v \frac{h(x)}{[1 - H(x)]^2} x dx + \frac{c}{k'} \right]$$

with initial condition $b(0) = 0$

$$b(v) = (1 - H(v)) \int_0^v \frac{h(x)}{[1 - H(x)]^2} x dx$$

using integration by part

$$\begin{aligned} b(v) &= (1 - H(v)) \left(\left[x \frac{1}{1 - H(x)} \right]_0^v - \int_0^v \frac{1}{1 - H(x)} dx \right) \\ &= (1 - H(v)) \left(v \frac{1}{1 - H(v)} - \int_0^v \frac{1}{1 - H(x)} dx \right) \\ &= v - \int_0^v \frac{1 - H(v)}{1 - H(x)} dx \end{aligned}$$

Now the last part is to show that $b(v)$ is indeed an equilibrium. In other words, there is no incentive to deviate from the strategy. Recall that the expected payoff of bidding b_1

$$\begin{aligned} \Pi(b_1, \hat{v_1}, v_1) &= \hat{v_1} + \int_0^{b^{-1}(b_1)} (v_1 - 2b(z)) h(z) dz - b_1 [1 - H(b^{-1}(b_1))] \\ &= \hat{v_1} + v_1 H(b^{-1}(b_1)) - \int_0^{b^{-1}(b_1)} 2b(z) h(z) dz - b_1 [1 - H(b^{-1}(b_1))] \end{aligned}$$

Using the bidding function from above

$$
\begin{aligned}
\int_0^{b^{-1}(b_1)} 2b(z)h(z)dz &= \int_0^{b^{-1}(b_1)} [2(1-H(z))h(z)(\int_0^z y\frac{h(y)}{(1-H(y))^2}dy)]dz \\
&= \left[-(1-H(z))^2\int_0^z y\frac{h(y)}{(1-H(y))^2}dy\right]_0^{b^{-1}(b_1)} \\
&\quad - \int_0^{b^{-1}(b_1)} \left(-(1-H(z))^2 z\frac{h(z)}{(1-H(z))^2}\right)dz \\
&= -(1-H(b^{-1}(b_1)))^2\int_0^{b^{-1}(b_1)} y\frac{h(y)}{(1-H(y))^2}dy + \int_0^{b^{-1}(b_1)} zh(z)dz \\
&= -[1-H(b^{-1}(b_1))]b_1 + \int_0^{b^{-1}(b_1)} zh(z)dz
\end{aligned}
$$

plug this in the expected utility

$$
\begin{aligned}
\Pi(b_1,\hat{v_1},v_1) &= \hat{v_1} + v_1H(b^{-1}(b_1)) - \int_0^{b^{-1}(b_1)} zh(z)dz \\
&= \hat{v_1} + \int_0^{b^{-1}(b_1)} (v_1 - z)h(z)dz
\end{aligned}
$$

This is clearly maximized at $b_1 = b(v_1)$

A.2 Lemma 1

With the merger, we know that the bid consortium bids according to the equilibrium bid $b^J(v_c)$. Since v_c is a random draw from $H(v)$, $b_M(v_c)$ is a random draw of $H((b^J)^{-1}(v))$. Let $G(v) = H((b^J)^{-1}(v))$. The expected payoff of bidder 1 given the bid consortium bid $b^J(v_c)$ is:

$$
\Pi(b_1;\hat{v}_1,v_1) = \hat{v}_1 + \int_0^{b_1}(v_1 - 2v_c)g(v_c)dv_c + (-b_1)(1-G(b_1))
$$

and the neccessary condition for the optimal bid $b(v_1)$ is

$$(v_1 - b)g(b) - [1 - G(b)] = 0$$

A.3 Example 2

The necessary condition for non-merger case is $(v-b)h(b)-[1-H(b)] = 2(v-b)(1-b)-(1-b)^2 \leq 0$. Since $(1-b) > 0$, the condition can be re-written $2(v-b)-(1-b) = 2v-b-1 \leq 0$. For $v < 0.5$, the inequality holds strictly, thus, $b_N(v) = 0$. For $v \geq 0.5$, the condition holds equality when $b = 2v-1$, thus, $b_N(v) = 2v-1$ (Note that the second derivate is $-2(v-b) < 0$, so it is indeed an optimal bid).

Let R_N denotes the revenue under non-merger.

$$R_N = \int_0^{\frac{1}{2}} \int_0^1 0h(v_l)h(v_1)dv_l dv_1 + \int_{\frac{1}{2}}^1 \int_0^{2v_1-1} 3v_l h(v_l)h(v_1)dv_l dv_1 + \int_{\frac{1}{2}}^1 \int_0^{2v_1-1} 3v_l h(v_l)h(v_1)dv_l dv_1$$
$$+ \int_{\frac{1}{2}}^1 \int_{2v_1-1}^1 3(2v_1-1)h(v_l)h(v_1)dv_l dv_1 = 0 + 0.075 + 0.075 = 0.15$$

Let R_M denotes the revenue under the merger. bid function $b_M(v) = v^2$ by (2). Let $\tilde{v}$ denotes the lowest value among all bidders. $\tilde{v}$ is a random variable drawn from $\tilde{H}(\tilde{v}) = 1-(1-F(\tilde{v}))^4$

$$R_M = \int_0^1 3\tilde{v}^2\tilde{h}(\tilde{v})d\tilde{v} = 0.2$$

A.4 Proposition 3

Recall that the necessary condition for $b_1^I(v)$ is $(v-b)h(b)-[1-H(b)] \leq 0$. Since the LHS is the marginal benefit of increasing bid, $\int_0^k (v-b)h(b)-[1-H(b)]db$ is total benefit of bidding $k \in [0,1]$. I will show the total benefit is always negative for $k > 0$ so any positive bidding is worse than bidding 0.

claim.

$$\int_0^k (v-b)h(b) - [1-H(b)]db \leq \int_0^k (v-b) - [1-b]db < 0$$

The second inequality is obvious. Since $-[1-H(b)] < -[1-b]$ by assumption, the only thing to show is $\int_0^k (v-b)h(b)db \leq \int_0^k (v-b)db$. Since $H(0)=0$ and $H(1)=1$, there is at least a point where $h(b)=1$ by mean value theorem on $[0,1]$. Suppose there are n points that satisfy $h(b)=1$ and denote each point $(c_1, c_2, ..., c_n)$. If there is a continuum of points that satisfy $h(b)=1$, I can safely ignore the continuum points since $\int (v-b)h(b)db = \int (v-b)db$ on the range. Then, I will show

$$\int_0^k (v-b)h(b)db \leq \int_0^k (v-b)db$$

If no crossing points occur before k, then it must $h(b) \leq 1$ for $b \in [0,k]$. If not, integrating both sides gives $H(k) > k$ and it violates the assumption. Now, suppose, without loss of generalization, n crossing points occur before k. Then, the inequality to show is

$$\int_0^{c_1} (v-b)h(b)db + ... + \int_{c_n}^k (v-b)h(b)db \leq \int_0^{c_1} (v-b)db + ... + \int_{c_n}^k (v-b)db$$
$$\int_0^{c_1} (v-b)(h(b)-1)db + ... + \int_{c_n}^k (v-b)(h(b)-1)db \leq 0$$

Each term in the second inequality is either positive or negative. Note that $(v-b)$ is positive since bidding above values is dominated. So the sign of each term is the sign of $h(b)-1$. First of all, the first term $\int_0^{c_1} (v-b)(h(b)-1)db$ is negative, otherwise, it implies $h(b)-1>0$ over $[0,c_1]$ and $H(c_1) > c_1$, which violate the assumption. Now if the second term is negative, the sum until the second term is negative. Even if the second term is positive, the sum until the second term must be negative. To show this, note that $\int_0^{c_1} (h(b)-1)db + \int_{c_1}^{c_2} (h(b)-1)db$ is negative. If it is positive, it implies $H(c_2) > c_2$ and violates the assumption. Thus, $|\int_0^{c_1}(h(b)-1)db| > |\int_{c_1}^{c_2}(h(b)-1)db|$. Then, since $(v-b)$ is positive and decreasing in b, $|\int_0^{c_1}(v-b)(h(b)-1)db| > |\int_0^{c_1}(v-b)(h(b)-1)db|$, which means the sum until the second term is negative. In this way, the sum up to any term in the LHS of second inequality is negative. (This statement is not precisely correct and should be rewritten.)

A.5 Example 3

When $H(x) = x^a$ and $a < 1$, the necessary condition for $b_N(v)$ is $(v-b)b^{a-1} - [1-b^a]$. Note that $\lim_{b\to 0+} h(b) = \lim_{b\to 0+} b^{a-1} = \infty$, thus $\lim_{b\to 0+}[(v-b)b^{a-1} - [1-b^a]] = \infty$. Also note that $\lim_{b\to v-}[(v-b)b^{a-1} - [1-b^a]] < 0$. Therefore, there must be $b^* \in (0, v)$ that satisfies $[(v-b^*)(b^*)^{a-1} - [1-(b^*)^a]] = 0$, which implies $b_M(v) > 0$.